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THE CONSTRUCTION OF SOLUTIONS OF PROBLEMS IN THE THEORY OF ELASTICITY IN THE FORM OF SERIES IN POWERS OF THE ELASTICITY CONSTANTS AND THEIR APPLICATION TO VISCOELASTICITY†

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New forms of writing well-known elasticity problems are proposed which enable algorithms to be obtained for constructing solutions in the form of series in powers of the elasticity constants. In particular, for a homogeneous isotropic body, the solutions are constructed in the form of series in powers of the Il'yushin parameter ω and the bulk modulus, while for a piecewise-homogeneous body, consisting of two materials, the solutions are constructed in the form of series in powers of ω_1 and ω_2. Applications of the forms of the solution obtained to problems of viscoelasticity are considered. Copyright © 1996 Elsevier Science Ltd.

The Volterra method and the method of integral transformations, which use a Laplace (or Laplace–Carson) transformation are widely employed to solve linear quasi-static problems of viscoelasticity [1–5]. In each of these the solution of the linear viscoelasticity problem is constructed from the solution of the corresponding elasticity problem. In the first case, the construction of the viscoelastic solution reduces to replacing the elasticity constants by Volterra operators and subsequent interpretation of the operator relations. In the second, it is required to obtain the originals from the known transforms. In both methods, the change from the elastic solution to the viscoelastic one is often accompanied by serious computational difficulties, connected with identifying the operator relations.

One of the possible ways of solving this problem is by representing the solution of the corresponding elasticity problem in a form convenient for subsequent identification. The first example of this kind is due to Volterra: for the problem of the deformation of a viscoelastic sphere with displacements specified on the surface he constructed a solution in the form of a series in positive powers of the time operator [6]. An effective method which enables the computational difficulties to be eliminated on changing from transforms to the originals was proposed by Il'yushin [7]. This method enabled an approximate solution of a wide range of problems of linear thermoviscoelasticity to be constructed.

The approach considered in this paper to the solution of problems of viscoelasticity belongs to this group of methods. The proposed algorithms for constructing elastic solutions, convenient for transferring to the corresponding viscoelastic problems, are very efficient as far as their numerical realization is concerned and they also enable solutions of new viscoelasticity problems to be constructed.

We will consider the method using the example of the solution of the problem of linear viscoelasticity for a homogeneous body made of a material that is hereditary-elastic for shear strains and elastic for bulk strains. Suppose that in the corresponding elasticity problem we use the Il'yushin parameter ω and the bulk modulus K as the elasticity constants. It is required to construct an elastic solution in displacements in the form of a series in integer powers of the Il'yushin parameter for a body of volume V, bounded by the surface $\Sigma = \Sigma_p + \Sigma_u$. Mass forces f_i action on the body, surface loads P_i are specified on the part Σ_p, and the displacements Φ_i are specified on the part Σ_u.

The components u_i of the required displacement vector must satisfy the Lamé equations

$$\frac{3}{2}\nabla^2 u_i + \left(\frac{1}{\omega_0} + \frac{1}{2}\right)\frac{\partial^2 u_j}{\partial x_i \partial x_j} = -\frac{1}{K\omega_0}\rho f_i - \left(\frac{\omega}{\omega_0} - 1\right)\left(\frac{3}{2}\nabla^2 u_i + \frac{1}{2}\frac{\partial^2 u_j}{\partial x_i \partial x_j}\right) \tag{1}$$

and the boundary conditions

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$$\mathbf{x} \in \Sigma_u: \quad u_i = \Phi_i \tag{2}$$

$$\mathbf{x} \in \Sigma_p: \left[\frac{3}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\frac{1}{\omega_0} - 1\right)\frac{\partial u_k}{\partial x_k}\delta_{ij}\right]\nu_j =$$

$$= \frac{P_i}{K\omega_0} - \left(\frac{\omega}{\omega_0} - 1\right)\left[\frac{3}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{\partial u_k}{\partial x_k}\delta_{ij}\right]\nu_j \tag{3}$$

Here ρ is the density, ν_j is the vector of the normal to the surface, ω_0 is a certain dimensionless number, δ_{ij} is the Kronecker delta, and x_i are Cartesian coordinates. Summation is carried out over repeated subscripts from 1 to 3.

Equations (1)–(3) are a consequence of identical transformations of the equations of the classical formulation of the problem of the theory of elasticity in displacements [8]. The solution of the problem will be sought in the form

$$u_i = \sum_{n=0}^{\infty}\left(\frac{\omega}{\omega_0} - 1\right)^n v_i^{(n)} \tag{4}$$

where $v_i^{(n)}$ are the required functions, which depend on the coordinates.

Substituting expansions (4) into (1)–(3) of the initial problem and comparing coefficients of like powers of $(\omega/\omega_0 - 1)$, we obtain the following recurrent series of boundary-value problems

$$\begin{aligned}
&\mathbf{x} \in V: \quad \frac{3}{2}\nabla^2 v_i^{(0)} + \left(\frac{1}{\omega_0} + \frac{1}{2}\right)\frac{\partial^2 v_j^{(0)}}{\partial x_i \partial x_j} = \frac{1}{K\omega_0}\rho f_i \\
&\mathbf{x} \in \Sigma_u: \quad v_i^{(0)} = \Phi_i \\
&\mathbf{x} \in \Sigma_p: \quad \left[\frac{3}{2}\left(\frac{\partial v_i^{(0)}}{\partial x_j} + \frac{\partial v_j^{(0)}}{\partial x_i}\right) + \left(\frac{1}{\omega_0} - 1\right)\frac{\partial v_k^{(0)}}{\partial x_k}\delta_{ij}\right]\nu_j = \frac{P_i}{K\omega_0} \\
&\qquad \cdots \\
&\mathbf{x} \in V: \quad \frac{3}{2}\nabla^2 v_i^{(n)} + \left(\frac{1}{\omega_0} + \frac{1}{2}\right)\frac{\partial^2 v_j^{(n)}}{\partial x_i \partial x_j} = -\left(\frac{3}{2}\nabla^2 v_i^{(n-1)} + \frac{1}{2}\frac{\partial^2 v_j^{(n-1)}}{\partial x_i \partial x_j}\right) \\
&\mathbf{x} \in \Sigma_u: \quad v_i^{(n)} = 0 \\
&\mathbf{x} \in \Sigma_p: \quad \left[\frac{3}{2}\left(\frac{\partial v_i^{(n)}}{\partial x_j} + \frac{\partial v_j^{(n)}}{\partial x_i}\right) + \left(\frac{1}{\omega_0} - 1\right)\frac{\partial v_k^{(n)}}{\partial x_k}\delta_{ij}\right]\nu_j = \\
&= -\left[\frac{3}{2}\left(\frac{\partial v_i^{(n-1)}}{\partial x_j} + \frac{\partial v_j^{(n-1)}}{\partial x_i}\right) - \frac{\partial v_k^{(n-1)}}{\partial x_k}\delta_{ij}\right]\nu_j
\end{aligned} \tag{5}$$

Solutions of the form (4) can be constructed, retaining both positive and negative powers of the parameter ω. We make the following substitution

$$\omega^N u_i = U_i \tag{6}$$

and multiply relations (1)–(3) by ω^N. The original problem then takes the form

$$\mathbf{x} \in V: \quad \frac{3}{2}\nabla^2 U_i + \left(\frac{1}{\omega_0} + \frac{1}{2}\right)\frac{\partial^2 U_j}{\partial x_i \partial x_j} + \frac{\omega^N}{K\omega_0}\rho f_i = -\left(\frac{\omega}{\omega_0} - 1\right)\left(\frac{3}{2}\nabla^2 U_i + \frac{1}{2}\frac{\partial^2 U_j}{\partial x_i \partial x_j}\right)$$

$$\mathbf{x} \in \Sigma_u: \quad U_i = \Phi_i \omega^N$$
$$\mathbf{x} \in \Sigma_p: \left[\frac{3}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) + \left(\frac{1}{\omega_0} - 1\right)\frac{\partial U_k}{\partial x_k}\delta_{ij}\right]\nu_j =$$
$$= \frac{P_i \omega^N}{K\omega_0} - \left(\frac{\omega}{\omega_0} - 1\right)\left[\frac{3}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{\partial U_k}{\partial x_k}\delta_{ij}\right]\nu_j \tag{7}$$

The solution of problem (7) will be sought in the form

$$U_i = \sum_{n=0}^{\infty}\left(\frac{\omega}{\omega_0} - 1\right)^n v_i^{(n)} \tag{8}$$

After expanding the factor ω^N in powers of $(\omega/\omega_0 - 1)$, substituting (8) into (7) and equating terms of like powers of the parameter ω, we obtain a series of boundary-value problems which differ from problems (5) solely by the right-hand sides in the first N approximations.

Reverting back to the original variable, we have the following form of the solution

$$u_i = \omega^{-N}\sum_{n=0}^{\infty}\left(\frac{\omega}{\omega_0} - 1\right)^n v_i^{(n)} \tag{9}$$

To prove the convergence of the method of constructing the solution of the elasticity problem in the form (9) and its numerical realization by effective computational procedures, in particular, the finite element method, we will use a variational formulation. The following variational equation

$$\langle u, \delta u\rangle - \int_V \frac{\rho f_i}{K\omega_0}\delta u_i dV - \int_{\Sigma_p} \frac{P_i}{K\omega_0}\delta u_i d\Sigma = -\left(\frac{\omega}{\omega_0} - 1\right)\langle u, \delta u\rangle_1 \tag{10}$$

and boundary conditions (2) correspond to boundary-value problem (1)–(3).

Here δu are virtual displacements, which satisfy the zero boundary conditions on Σ_u, and we have also used the following notation

$$\langle u,v\rangle = \int_V \left[\frac{3}{2}\left(\frac{\partial u_i}{\delta x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\frac{1}{\omega_0} - 1\right)\frac{\partial u_k}{\partial x_k}\delta_{ij}\right]\frac{\partial v_i}{\partial x_j} dV$$
$$\langle u,v\rangle_1 = \int_V \left[\frac{3}{2}\left(\frac{\partial u_i}{\delta x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{\partial u_k}{\partial x_k}\delta_{ij}\right]\frac{\partial v_i}{\partial x_j} dV \tag{11}$$
$$\langle u,v\rangle_2 = \int_V \frac{1}{\omega_0}\frac{\partial u_k}{\partial x_k}\delta_{ij}\frac{\partial v_i}{\partial x_j} dV$$

The following relation holds for expressions (11)

$$\langle u, v\rangle = \langle u, v\rangle_1 + \langle u, v\rangle_2$$

The solution of the variational problem will again be sought in the form of expansion (4). To find the coefficients of this series we have the following recurrent sequences of variational problems

$$\langle v^{(0)}, \delta u\rangle - \int_V \frac{\rho f_i}{K\omega_0}\delta u_i dV - \int_{\Sigma_p} \frac{P_i}{K\omega_0}\delta u_i d\Sigma = 0$$
$$\ldots$$
$$\langle v^{(n)}, \delta u\rangle = -\langle v^{(n-1)}, \delta u\rangle_1 \tag{12}$$

The convergence of the proposed procedure can be proved as follows: In view of the arbitrary nature of the variations δu_i we will initially put $\delta u_i = v_i^n - v_i^{(n-1)}$ and then $\delta u_i = v_i^{(n)} + v_i^{(n-1)}$, and add the results.

After some identical transformations and taking into account the symmetry of the scalar products (11) we obtain

$$\langle v^{(n)}, v^{(n)}\rangle - \langle v^{(n-1)}, v^{(n-1)}\rangle = -\langle v^{(n)} + v^{(n-1)}, v^{(n)} + v^{(n-1)}\rangle_1 - \langle v^{(n)}, v^{(n)}\rangle_2 - \langle v^{(n-1)}, v^{(n-1)}\rangle_2$$

From the fact that the scalar squares are positive definite it follows that the right-hand side of the last equation is negative. Hence, the norms of the coefficients of series (4) decreases as n increases, and, consequently, series (4) converges with respect to the norm $\| u \| = \langle u, u\rangle^{1/2}$ when the following condition is satisfied

$$0 < \omega < 2\omega_0 \tag{13}$$

We will consider the proposed method as it applies to the problem for a piecewise-homogeneous body, consisting of different materials, that are hereditarily elastic for shear strains and elastic for bulk strains. Thus, we have two viscoelastic bodies occupying volumes V_1 and V_2 and bounded by the surfaces $\Sigma_1 = \Sigma_{p1} + \Sigma_{u1} + \Sigma_{12}$, $\Sigma_2 = \Sigma_{p2} + \Sigma_{u2} + \Sigma_{12}$. The bodies are in contact over the surface Σ_{12}. Mass forces f_{i1} are applied to the first of these, surface loads P_{i1} are specified on the part of the surface Σ_{pi}, and displacements Φ_{i1} are specified on the surface Σ_{u1}. The other body is subjected to similar forces. The conditions of continuity of the strains and stresses, normal and tangential to the contact surface, are satisfied on the contact boundary.

We propose to construct the solution of the corresponding elasticity problem in the form

$$u_{i1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\omega_1}{\omega_{01}} - 1\right)^m\left(\frac{\omega_2}{\omega_{02}} - 1\right)^n v_{i1}^{(m,n)} \tag{14}$$

$$u_{i2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\omega_1}{\omega_{01}} - 1\right)^m\left(\frac{\omega_2}{\omega_{02}} - 1\right)^n v_{i2}^{(m,n)}$$

(the subscripts 1 and 2 relate to volumes V_1 and V_2, respectively).

By analogy with the problem considered above, to find the coefficients $v_{ik}^{(m,n)}$ $(k = 1, 2)$ of series (14), the following recurrent sequence of variational problems can be obtained

$$\langle v^{(0,0)}, \delta u\rangle^1 + \langle v^{(0,0)}, \delta u\rangle^2 = \int_{V_1} \frac{\rho_1 f_{i1}}{K_1 K_2 \omega_{01}\omega_{02}}\delta u_i dV +$$

$$+\int_{V_2} \frac{\rho_2 f_{i2}}{K_1 K_2 \omega_{01}\omega_{02}}\delta u_i dV + \int_{\Sigma_{P1}} \frac{P_{i1}\delta u_i}{K_1 K_2 \omega_{01}\omega_{02}} d\Sigma + \int_{\Sigma_{P2}} \frac{P_{i2}\delta u_i}{K_1 K_2 \omega_{01}\omega_{02}} d\Sigma$$

$$\ldots$$

$$\langle v^{(m,n)}, \delta u\rangle^1 + \langle v^{(m,n)}, \delta u\rangle^2 = \langle v^{(m-1,n)}, \delta u\rangle_1^1 - \langle v^{(m,n-1)}, \delta u\rangle_1^2 \tag{15}$$

Here

$$\langle u,v\rangle^1 = \int_{V_1} \frac{1}{K_2\omega_{02}}\left[\frac{3}{2}\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i}\right) + \left(\frac{1}{\omega_{01}} - 1\right)\frac{\partial u_{k1}}{\partial x_k}\delta_{ij}\right]\frac{\partial v_i}{\partial x_j} dV$$

$$\langle u,v\rangle_1^1 = \int_{V_1} \frac{1}{K_2\omega_{02}}\left[\frac{3}{2}\left(\frac{\partial u_{i1}}{\partial x_j} + \frac{\partial u_{j1}}{\partial x_i}\right) - \frac{\partial u_{k1}}{\partial x_k}\delta_{ij}\right]\frac{\partial v_i}{\partial x_j} dV$$

$$\langle u,v\rangle_2^1 = \int_{V_1} \frac{1}{K_2\omega_{01}}\frac{\partial u_{k1}}{\partial x_k}\delta_{ij}\frac{\partial v_i}{\partial x_j} dV$$

$$\langle u,v\rangle^2 = \int_{V_2} \frac{1}{K_1\omega_{01}}\left[\frac{3}{2}\left(\frac{\partial u_{i2}}{\partial x_j} + \frac{\partial u_{j2}}{\partial x_i}\right) + \left(\frac{1}{\omega_{02}} - 1\right)\frac{\partial u_{k2}}{\partial x_k}\delta_{ij}\right]\frac{\partial v_i}{\partial x_j} dV$$

$$\langle u,v\rangle_1^2 = \int_{V_2} \frac{1}{K_1\omega_{01}}\left[\frac{3}{2}\left(\frac{\partial u_{i2}}{\partial x_j} + \frac{\partial u_{j2}}{\partial x_i}\right) - \frac{\partial u_{k2}}{\partial x_k}\delta_{ij}\right]\frac{\partial v_i}{\partial x_j} dV$$

$$\langle u,v\rangle_2^2 = \int_{V_2} \frac{1}{K_2\omega_{01}\omega_{02}}\frac{\partial u_{k2}}{\partial x_k}\delta_{ij}\frac{\partial v_i}{\partial x_j}dV$$

Using the version of the proof of convergence considered earlier, it can be shown that in this problem series (14) will converge when the following conditions are satisfied

$$0<\omega_1<2\omega_{01},\quad 0<\omega_2<2\omega_{02} \tag{16}$$

To construct solutions relating both positive and negative powers of ω_k we will use the following change of variables

$$U_{i1}=\omega_1^M\omega_2^N u_{i1},\quad U_{i2}=\omega_1^M\omega_2^N u_{i2} \tag{17}$$

Equations (15) are then multiplied by ω_1^M, ω_2^N, change (17) is made, the factor ω_1^M, ω_2^N is expanded in series in powers of $(\omega_1/\omega_{01}-1)(\omega_2/\omega_{02}-1)$, and the solution in the new variables U_{ik} is sought in the form of series (14). The recurrent series of problems in the coefficients of these series will differ from Eqs (15) solely in the right-hand sides in the first $M\times N$ approximations. Reverting to the initial variables, we obtain the solution in the form of expansions in positive and negative powers of ω_k.

The construction of solutions of problems of the theory of elasticity in the form of a series in powers of two elastic constants also enables us to use Volterra's method effectively in problems of linear viscoelasticity for a homogeneous body in which the hereditarily elasticity properties of the material manifest themselves both in shear and bulk strains.

For a homogeneous body, when using the shear modulus G and the bulk modulus K as the elasticity constants, the solution is constructed in the form

$$u_i=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{G}{G_0}-1\right)^m\left(\frac{K}{K_0}-1\right)^n v_i^{(m,n)} \tag{18}$$

The recurrent sequence of variational problems for finding the coefficients of series (18) has the form

$$\langle v^{(0,0)},\ \delta u\rangle=\int_V\frac{\rho f_i}{G_0K_0}\delta u_i dV+\int_{\Sigma_p}\frac{P_i}{G_0K_0}\delta u_i d\Sigma$$

$$\dots$$

$$\langle v^{(m,n)},\ \delta u\rangle=-\langle v^{(m-1,n)},\ \delta u\rangle_1-\langle v^{(m,n-1)},\ \delta u\rangle_2 \tag{19}$$

Here

$$\langle u,v\rangle=\int_V\left[\frac{1}{K_0}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)+\frac{1}{G_0}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right]\frac{\partial v_i}{\partial x_j}dV$$

$$\langle u,v\rangle_1=\int_V\frac{1}{K_0}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\frac{\partial v_i}{\partial x_j}dV \tag{20}$$

$$\langle u,v\rangle_2=\int_V\frac{1}{G_0}\frac{\partial u_k}{\partial x_k}\delta_{ij}\frac{\partial v_i}{\partial x_j}dV$$

The solution in the form of series (18) converges provided that

$$0<G<2G_0,\quad 0<K<2K_0 \tag{21}$$

To construct solutions containing both positive and negative powers of G and K we will make the following change of variables

$$U_i=G^MK^Nu_i$$

We propose to use the finite element method to realize this method numerically. In this connection we need to make the following important observation: the recurrent sequences of problems obtained for finding solutions in the form of series in powers of the elasticity constants have the same expressions for the left-hand sides of the equations and, consequently, the algebraic analogues to which they converge when using the finite element method also differ solely in their right-hand sides. Practical applications of the method have shown that the computer time required to construct a solution of each individual successive problem from the general recurrent sequence is 15–20 times less than that required to construct a solution of the first problem of the sequence considered. This fact distinguishes the proposed algorithms as the most efficient for numerical realization on a computer among the various other methods of solving problems of the theory of viscoelasticity based on Volterra's method and the method of successive transformations.

Example 1. Consider a short hollow cylinder occupying the region $a \leqslant r \leqslant b, 0 \leqslant z \leqslant r\, 2L$ in a cylindrical system of coordinates. The outer surface $r = b$ is fixed, the inner surface $r = a$ and the ends $z = 0$ and $z = 2L$ are stress-free. Axial mass forces act on the cylinder. It is required to determine the elastic axisymmetric displacement field.

Initially, using the finite-difference method we solve the problem of the elastic cylinder for $a/b = 0.4$; $2L/b = 1$; $\omega = 0.069$. We use this solution as a test.

We then consider the variational problem (12), the solution of which we construct in the form of series (9). The problem is solved numerically using the finite element method. Here the type of elements and the degree of discretization are similar to the test version.

The number of terms of series (9) is chosen so that the quantity

$$\varepsilon = \max_{V} \frac{|\mathbf{u}^{(n)} - \mathbf{u}^{(n-1)}|}{|\mathbf{u}^{(n)}|}$$

becomes less than a previously specified small quantity ε_*.

Calculations showed that when the number of terms of the series is increased by a factor of no more than 2 the accuracy increases by a factor 10. The dependence of the number of terms on ω_0 and on N for a specified accuracy $\varepsilon_* = 0.01$ is shown in Table 1. The numerator shows the number of iterations and the denominator shows the value of the relative error ε as a percentage.

The solution obtained in the form of series (9) was compared with the test solution at all the nodal points of the region calculated. Table 1 shows the relative error of the solution as a function of ω_0 and N. Here the number ω_0 satisfied condition (13). The version when condition (13) is not satisfied ($\omega_0 = 0.02$) is also given. In this case series (9) was divergent.

Example 2. We will consider the model problem of the stress–strain state of a double-layer viscoelastic cylinder having the following dimensions: $\alpha = a/b = 0.3$, $\beta = c/b = 1.1$, $\gamma = 2L/b = 2$, where a is the inner radius, b is the radius of the contact surface, c is the outer radius and $2L$ is the length of the cylinder. A constant pressure P acts on the inner surface of the cylinder, and the outer and end surfaces are assumed to be stress-free. The materials of the layers have different mechanical characteristics, heredity-elastic for shear and elastic for bulk strains. The inner layer is given the index $k = 1$ and the outer layer is given the index $k = 2$.

Using the above method we constructed a solution of the corresponding elasticity problem retaining positive and negative powers of the Il'yushin parameter ω_k. Taking the form of this solution into account, and using Volterra's method, the viscoelastic solution can be written as follows:

$$\mathbf{u}_k(\mathbf{x},t) = \bar{\omega}_1^{-M} \bar{\omega}_2^{-N} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{\bar{\omega}_1}{\omega_{01}} - 1 \right)^m \left(\frac{\bar{\omega}_2}{\omega_{02}} - 1 \right)^n \mathbf{v}_k^{(m,n)}(\mathbf{x}) g(t) \tag{22}$$

Table 1

ω_0	$N = 0$	1	2	3
0.04	18/0.22	6/0.23	5/0.02	5/0.0
0.143	7/0.95	5/0.11	5/0.02	6/0.02
0.02	10/1.34	6/0.21	7/0.02	8/0.02
0.05	20/4.84	11/0.55	12/0.15	14/0.12

Table 2

Version	Version							
	1	2	3	4	5	6	7	8
A_1	3.0	0.3	0.4	0.04	3.0	0.3	0.4	0.04
A_2	4.0	0.4	4.0	0.4	4.0	0.4	4.0	0.4
β_1	0.3	0.3	0.3	0.3	0.4	0.4	0.4	0.4
β_2	0.4	0.4	0.4	0.4	4.0	4.0	4.0	4.0

where $g(t)$ is a function of time, to which all the external forces are proportional.

The calculation of relations (22) reduces to identifying the product of the operators of the following structure

$$I_{m,n} = \tilde{\omega}_1^{-M} \tilde{\omega}_2^{-N} \left(\frac{\tilde{\omega}_1}{\omega_{01}} - 1 \right)^m \left(\frac{\tilde{\omega}_2}{\omega_{02}} - 1 \right)^n g(t) \tag{23}$$

where $\tilde{\omega}_k$ are integral Volterra operators

$$\tilde{\omega}_k(g) = \omega_k \left[g(t) - \int_0^t R_k(t-\tau) g(\tau) d\tau \right]$$

with relaxation kernels $R_k(t)$.

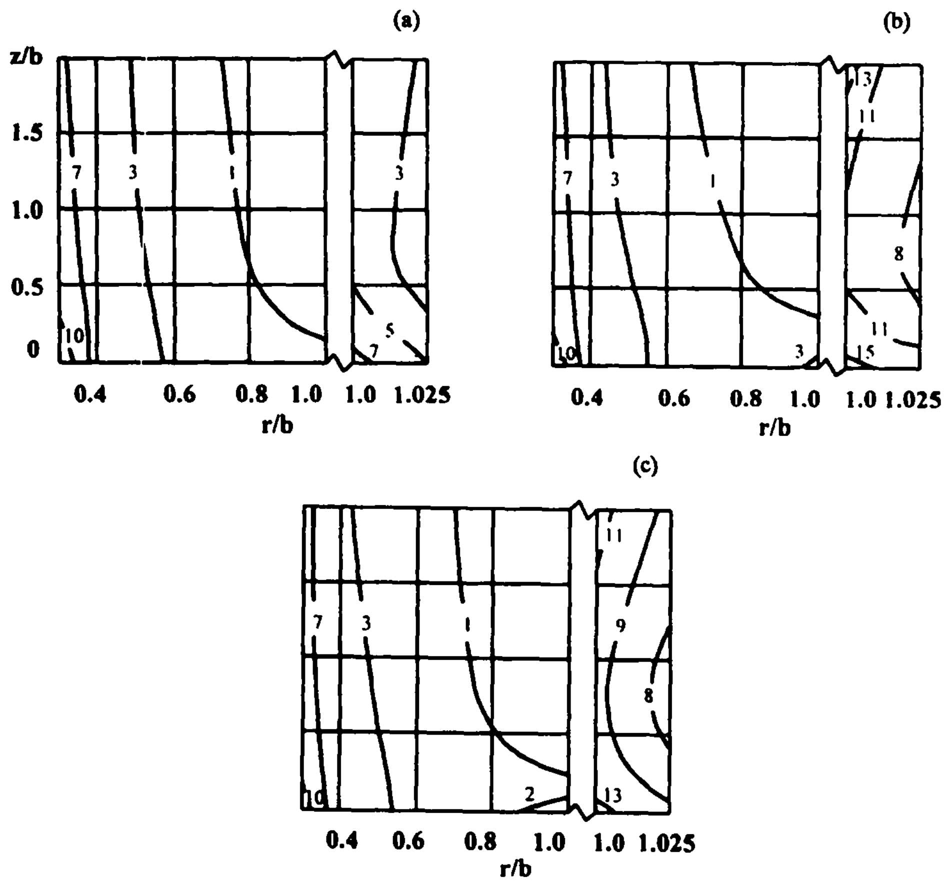

Fig. 1.

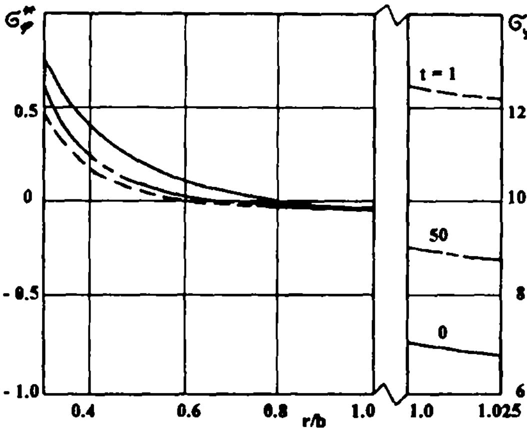

Fig. 2.

When the function $g(t)$ is represented in the form of a section of a power series with respect to time, analytic expressions are obtained for calculating relations (23) using in the operators $\tilde{\omega}_k$ kernels of an exponential type, a power type and an Abel type. For the Rzhanitsyn kernel $R_k = A_k \exp(-\beta_k t) t^{\alpha_k - 1}$ analytic formulae can be obtained for $\beta_1 = \beta_2$. The expressions for $\tilde{\omega}_1$ and $\tilde{\omega}_3$ may contain a different type of kernel. For certain combinations of kernels one can also obtain analytic expressions for identifying the operator expression (23), for example, for exponential and power kernels. The analytic expressions obtained are given in [9].

When the operation of identifying the operator relations (23) cannot be carried out analytically, one can use the method of quasi-constant operators [10]. A method of using this approach for the proposed method was described in [11] and it was shown that it gives satisfactory accuracy.

In the example considered we used an exponential-type relaxation kernel

$$R_k(t) = A_k \exp(-\beta_k t)$$

Calculations were carried out for instantaneous values of the Il'yushin parameters $\omega_1 = 0.222$ and $\omega_2 = 0.30778$ and bulk moduli $K_1 G_1 = 3, K_2/G_1 = 216.7$ (G_1 is the instantaneous shear modulus of the inner layer of the cylinder). Versions of the values of the constants A_k and β_k of the kernels $R_k(t)$ considered are shown in Table 2. The values were chosen from the following model considerations. For $A_k = 3$ and $\beta_k = 4$ the ratio of the instantaneous value of the Il'yushin parameter ω_k to its long-term value ω_k^∞ was equal to four, and the ratio

$$\varkappa = \left[\int_0^t R_k(t-\tau)d\tau - \int_0^\infty R_k(t-\tau)d\tau\right]\left[\int_0^\infty R_k(t-\tau)d\tau\right]^{-1}$$

became less than 3% for $t = 1$, i.e. it was assumed that by this time rheological processes in the material had practically ended. A material with these parameters is called a material with pronounced rheological properties, which manifest themselves rapidly with time.

For $A_k = 0.3$ and $\beta_k = 0.4$ the ratio ω_k/ω_k^∞ remained unchanged, but the value of $\varkappa$ reached a value of 3% at the instant $t = 10$. A material corresponding to these parameters has been called a material with pronounced rheological properties, which manifest themselves slowly with time. The parameters $A_k = 0.4$ and $\beta_k = 4$ define a medium with only weak rheological properties ($\omega_k/\omega_k^\infty = 1.11$), which manifest themselves rapidly with time, while the parameters $A_k = 0.04$ and $\beta_k = 0.4$ define a medium with only slight rheological properties which manifest themselves slowly with time.

An analysis of the results obtained enabled us to clarify how the stresses vary with time in a composite viscoelastic cylinder. The most interesting case is the one in which one of the layers consists of a material with pronounced rheological properties which manifest themselves rapidly with time (in the version considered this is the inner layer), while the second layer is made of material with the same characteristics but which manifest themselves slowly with time.

The lines of equal level of intensity of shear stresses $\sigma^* = \sigma/P$ shown in Fig. 1 for the lower half of the cylinder at the instants of time $t = 0$ (a), $t = 1$ (b) and $t = 50$ (c), correspond to this version of the parameters. The scale divisions within the layer are constant. In the first layer the value $\sigma^* = 0.12$ corresponds to the first level and the value $\sigma^* = 0.83$ corresponds to the tenth level. In the second layer, on the first level $\sigma^* = 1.04$ and on the fifteenth level $\sigma^* = 2.56$. It can be seen by comparing Fig. 1(a)–(c) that for an external load which is constant in time, the stresses on certain parts of the region investigated vary non-monotonically with time. When the thickness of the outer layer is reduced, this mechanical effect becomes more striking. This is confirmed by Fig. 2 where we show the stress distribution $\sigma_\varphi^* = \sigma_\varphi/P$ over the thickness of the cylinder at different instants of time.

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